Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 269 (2004) 535–548
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On acoustic radiation by a rigid object in a fluid flow

A. Zinoviev*, D.A. Bies

School of Mechanical Engineering, The University of Adelaide, North Terrace, Adelaide 5005, Australia

Received 26 June 2002; accepted 2 January 2003

Abstract

A problem of sound radiation by an absolutely rigid object, moving with respect to the surrounding fluid, is considered on the basis of the Lighthill's equation for aerodynamic sound. An integral representation of the radiated acoustic field is utilized, where the field is characterized as the sum of three fields, generated by a volume distribution of monopoles and by distributions of monopoles and dipoles on the surface of the rigid object. It is shown that, due to a discontinuity of Lighthill's stress tensor on the rigid boundary, a layer of surface divergence of hydrodynamic stresses on the boundary must be taken into account when evaluating the volume integral over Lighthill's quadrupole sources. When the contribution of the surface divergence is included in the solution of Lighthill's equation, amplitudes of the monopole and dipole sound radiated by the rigid object are shown to depend on the potential components of the normal velocity and the pressure on the rigid surface. The obtained solution is compared with Curle's solution for this problem, which establishes that the sound radiation by a rigid object is determined by the force exerted by the object upon the fluid. Both solutions are applied to two known problems of sound scattering and radiation by a rigid sphere in variable pressure and velocity fields. It is shown that predictions based on the obtained solution are equivalent to the results known from literature, whereas Curle's solution gives predictions contradicting the known results. It is also shown that the Ffowcs Williams and Hawkings equation, which coincides with Curle's equation for an immoveable rigid object, does not lead to the correct predictions as well.

1. Introduction

The problem of sound radiation by a rigid object in a fluid flow is one of central importance in aeroacoustics. A significant success in solving this problem was achieved in 1952, when Lighthill published his well-known work [1], where he derived an equation which determines that sound

*Corresponding author.
E-mail address: alexei.zinoviev@adelaide.edu.au (A. Zinoviev).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00021-X

radiated by turbulent flow in a fluid without solid boundaries has quadrupole characteristics. In 1955 Curle [2] extended Lighthill's theory to include flow with solid boundaries and showed that an immoveable solid object in a turbulent flow radiates dipole sound. According to Curle, the amplitude of the dipole sound is determined by a distribution of the total force per unit area over the solid boundaries. For an acoustically compact object Curle's theory establishes that the dipole sound amplitude is proportional to the total force acting upon the fluid by the rigid boundaries. Curle's theory is often used in the form of Ffowcs Williams–Hawkings equation [3], which takes into account motion of the rigid object.

This article is devoted to the problem of sound radiation by a rigid object in a fluid. The necessity to revisit this problem was determined by two factors. First, experimental data obtained by the second author and his co-authors [4] to verify Curle's solution did not converge to Curle's prediction; and second, a recently published article [5] argues that the present aeroacoustic theory contains some fundamental flaws. Thus, the purpose of the current article is to analyze the problem of sound radiation by a rigid object and to compare the solution so obtained with the known solutions for this problem provided by Curle [2] and Ffowcs Williams and Hawkings [3].

The authors realize that due to the wide acceptance of the Curle–Ffowcs Williams–Hawkings theory, some readers may consider the conclusions of the current article controversial. Nevertheless, experimental evidence would suggest that the problem of sound generation by a solid object in fluid flow would benefit from some reconsideration.

2. Lighthill's theory of aerodynamic sound

Lighthill [1] considered the general problem of sound generation and propagation in a uniform fluid medium. He showed that the equations of continuity and momentum for such a medium could be reduced to the following equation with respect to the fluid density, ρ:

$$\rho'(\mathbf{x}, t) = \rho - \rho_0 = \frac{1}{4\pi c_0^2} \int\int\int_{V_{tot}} \frac{1}{|\mathbf{x}-\mathbf{y}|} \frac{\partial^2}{\partial y_i \partial y_j} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) \mathrm{d}\mathbf{y}, \tag{1}$$

where T_{ij} is Lighthill's stress tensor, given by

$$T_{ij} = \rho v_i v_j + p_{ij} - c_0^2 \rho \delta_{ij}, \quad i, j = 1,\ 2,\ 3. \tag{2}$$

In Eqs. (1) and (2) c_0 is the speed of sound in the fluid at rest, v_i is the ith component of the velocity of the fluid, δ_{ij} is Kronecker's delta function, ρ_0 is the density of the fluid at equilibrium, $\mathbf{x} = (x_1,\ x_2,\ x_3)$ is the co-ordinate of the observation point, $\mathbf{y} = (y_1,\ y_2,\ y_3)$ is the co-ordinate of the source point, tensor p_{ij} is the stress tensor and the integral is taken over the total volume, V_{tot}, of the fluid. Indices repeated in a single term are to be summed from 1 to 3. For example, $\partial v_k/\partial x_k$ must be understood as $\partial v_1/\partial x_1 + \partial v_2/\partial x_2 + \partial v_3/\partial x_3$.

Lighthill demonstrated that Eq. (1) could be reduced to the following equation:

$$\rho'(\mathbf{x},\ t) = \rho - \rho_0 = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int\int\int_{V_{tot}} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) \frac{1}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\mathbf{y}, \tag{3}$$

which represents the well-known idea of Lighthill that aerodynamic sound radiated by a turbulent flow in a fluid without boundaries has quadrupole characteristics and the amplitude of the quadrupole sources is proportional to the value of the tensor T_{ij}.

3. Application of Lighthill's theory to a fluid with solid boundaries

3.1. Spatial layout

Let the problem of sound radiation by a rigid object be solved for a fluid with the spatial layout shown in Fig 1. It is assumed that there is only one solid object of boundary, S, which has no sharp edges. The surface, S_1, at which $T_{ij} = 0$, encloses the boundary S. The volume, V_0, bounded by S and S_1, includes all regions in the fluid where $T_{ij} \neq 0$. It is assumed that $T_{ij} \neq 0$ at S. The object is stationary with respect to the observer, but relative motion of the object and the surrounding fluid is allowed. The observation point, **x**, lies outside the volume V_0, while the integration point, **y**, is within the volume V_0.

3.2. Formulation of the solution

In considering the influence of solid boundaries on sound generation by a solid object in a fluid flow, the acoustic field radiated by the flow is represented as a sum of Lighthill's solution (1) and an integral over the solid boundary, S:

$$\rho'(\mathbf{x},\ t) = \frac{1}{4\pi c_0^2} \int\int\int_{V_{tot}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \frac{\mathrm{d}\mathbf{y}}{r} + \frac{1}{4\pi} \int\int_S \left(\frac{1}{r}\frac{\partial \rho'}{\partial n} + \frac{1}{r^2}\frac{\partial r}{\partial n}\rho' + \frac{1}{c_0 r}\frac{\partial r}{\partial n}\frac{\partial \rho'}{\partial t} \right) \mathrm{d}S(\mathbf{y}), \quad (4)$$

where $r = |\mathbf{x} - \mathbf{y}|$ and **n** is the outward normal from the fluid.

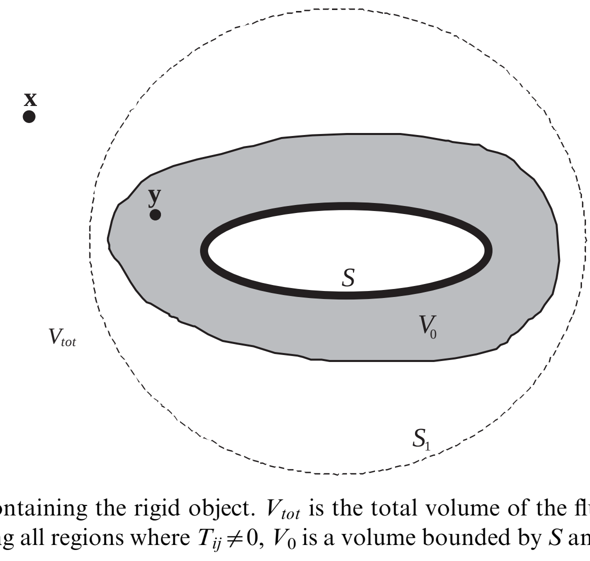

Fig. 1. Layout of the fluid containing the rigid object. V_{tot} is the total volume of the fluid, S is the surface of the rigid object, S_1 is a surface enclosing all regions where $T_{ij} \neq 0$, V_0 is a volume bounded by S and S_1, **x** is the observation point, **y** is the source point.

After introducing the integral formulation (4), the next step in solving the problem of sound radiation by a rigid object is to transform the volume integral into a more convenient form by replacing the derivatives over the source point, **y**, with derivatives over the observation point, **x**. To transform Eq. (1) to Eq. (3), Lighthill [1] considered the source distribution $\partial^2 T_{ij}/\partial y_i \partial y_j$ in detail. At the same time, he mentioned that the same result could be achieved by using the divergence theorem twice. The latter procedure was used by Curle [2] and is utilized here also.

Before the divergence theorem can be applied, the following transformations of volume integrals must be carried out:

$$\int\int\int_{V_{tot}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \frac{\mathrm{d}\mathbf{y}}{r} = \int\int\int_{V_{tot}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_i}\int\int\int_{V_{tot}} \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}\mathbf{y}}{r}, \tag{5}$$

$$\int\int\int_{V_{tot}} \frac{\partial T_{ij}}{\partial y_j} \frac{\mathrm{d}\mathbf{y}}{r} = \int\int\int_{V_{tot}} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_j}\int\int\int_{V_{tot}} T_{ij}\frac{\mathrm{d}\mathbf{y}}{r}. \tag{6}$$

These equations can be derived easily by differentiating the first terms on the right-hand parts as a product of two functions, while bearing in mind that $r = |\mathbf{x} - \mathbf{y}|$ and $\partial r/\partial x_i = -\partial r/\partial y_i$.

The first integrals on the right-hand of Eqs. (5) and (6) are to be transformed here using the divergence theorem. This theorem is one of the foundations of vector analysis and can be found in many textbooks; for example, in Refs. [6,7]. However, due to the importance of the divergence theorem for the present analysis, it is useful to introduce the theorem here.

3.3. Formulation of the divergence theorem

The divergence theorem states that the integral of the flux out of a closed surface, S_V, of a vector field, $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x_1, x_2, x_3) = (F_1, F_2, F_3)$, is equal to the volume integral of the divergence of the field, **F**, over the volume, V, enclosed by S_V [7]. For a bounded closed region, V, whose boundary, S_V, is a closed regular surface, and for continuously differentiable functions F_i, $i = 1, 2, 3$, the divergence theorem is formulated as follows [6]:

$$\int\int\int_V \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right) \mathrm{d}V = \int\int_{S_V} (F_1 l_1 + F_2 l_2 + F_3 l_3)\, \mathrm{d}S_V, \tag{7}$$

or, equally, as

$$\int\int\int_V \nabla \cdot \mathbf{F}(\mathbf{x})\, \mathrm{d}V = \int\int_{S_V} \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}\, \mathrm{d}S_V, \tag{8}$$

where $\mathbf{n} = (l_1, l_2, l_3)$ is the *outward* normal to the surface S_V.

An essential idea for this analysis is that the divergence theorem (7) can be extended to the case where the functions F_i are not continuous on the boundary S_V [8]. Let F_i and their derivatives be continuous on both sides of S_V. Simultaneously let F_i have a discontinuity at S_V, so that limiting values, $F_i(\mathbf{x}^{(+)})$ and $F_i(\mathbf{x}^{(-)})$, exist on the positive and negative side of S_V, respectively.

In this case, the notion of the surface divergence $\nabla_{S_V} \cdot \mathbf{F}$ of the vector field $\mathbf{F} = (F_1, F_2, F_3)$ may be introduced:

$$\nabla_{S_V} \cdot \mathbf{F}(\mathbf{x}) = \mathbf{n} \cdot \left[\mathbf{F}\left(\mathbf{x}^{(+)}\right) - \mathbf{F}\left(\mathbf{x}^{(-)}\right)\right]. \tag{9}$$

For the discontinuous vector field $\mathbf{F}$, the divergence theorem holds if $\nabla \cdot \mathbf{F}$ in Eq. (8) is replaced with its surface analogue, $\nabla_{S_V} \cdot \mathbf{F}$, at the surface S_V [8]. Then the divergence theorem can be formulated as follows:

$$\int\int\int_V \nabla \cdot \mathbf{F}(\mathbf{x})\, \mathrm{d}V = \int\int_{S_V} \mathbf{F}\left(\mathbf{x}^{(-)}\right) \cdot \mathbf{n}\, \mathrm{d}S + \int\int_{S_V} \nabla_{S_V} \cdot \mathbf{F}(\mathbf{x})\, \mathrm{d}S. \tag{10}$$

3.4. The theory of Curle–Ffowcs Williams–Hawkings

According to Lighthill [1], in formulas (5) and (6) the integration is carried out over the total volume, V_{tot}, of the fluid. In all practical tasks, however, the integration can be done over any finite volume, which includes all regions in the fluid where $T_{ij} \neq 0$. Curle [2] used the volume V_0 (Fig. 1) as the volume of integration to obtain the following volume integrals:

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int\int_{V_0} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y}, \tag{11}$$

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int\int_{V_0} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y}, \tag{12}$$

which he transformed by means of the divergence theorem into surface integrals over bounding surfaces S and S_1:

$$\int\int\int_{V_0} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int_S l_i \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}S(\mathbf{y})}{r} + \int\int_{S_1} l_i \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}S(\mathbf{y})}{r}, \tag{13}$$

$$\int\int\int_{V_0} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int_S l_j T_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r} + \int\int_{S_1} l_j T_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r}. \tag{14}$$

Indeed, Eqs. (13) and (14) can be reduced to the divergence theorem (7) for vector fields, $\mathbf{F}_1 = (F_{11}, F_{12}, F_{13})$ and $\mathbf{F}_2 = (F_{21}, F_{22}, F_{23})$, determined by

$$F_{11} = \frac{\partial T_{1j}}{\partial y_j}\frac{1}{r}, \quad F_{12} = \frac{\partial T_{2j}}{\partial y_j}\frac{1}{r}, \quad F_{13} = \frac{\partial T_{3j}}{\partial y_j}\frac{1}{r}, \tag{15}$$

$$F_{21} = T_{i1}\frac{1}{r}, \quad F_{22} = T_{i2}\frac{1}{r}, \quad F_{23} = T_{i3}\frac{1}{r}. \tag{16}$$

As T_{ij} and its derivatives are equal to zero on the surface S_1, Eqs. (13) and (14) become:

$$\int\int\int_{V_0} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int_S l_i \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}S(\mathbf{y})}{r}, \tag{17}$$

$$\int\int\int_{V_0} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int_S l_j T_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r}, \tag{18}$$

and, due to Eqs. (11) and (12):

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int_S l_i \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}S(\mathbf{y})}{r}, \tag{19}$$

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \int\int_S l_j T_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r}. \tag{20}$$

Substituting Eqs. (19) and (20) into Eqs. (5) and (6), it is possible to obtain the following representation of the integral over Lighthill's sources:

$$\int\int\int_{V_{tot}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\frac{\mathrm{d}\mathbf{y}}{r} = \frac{\partial^2}{\partial x_i \partial x_j}\int\int\int_{V_{tot}} \frac{T_{ij}}{r}\mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_i}\int\int_S l_j T_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r} + \int\int_S l_i \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}S(\mathbf{y})}{r}. \tag{21}$$

The first term in Eq. (21) is Lighthill's integral (3), which determines the quadrupole sound produced by turbulence in the absence of rigid objects. The last two terms in Eq. (21) can be understood as contributions of layers of dipole and monopole sources on the rigid boundary.

If Eq. (21) is substituted into the general solution (4), one can obtain the following equation, derived by Curle [2]:

$$4\pi c_0^2 \rho'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j}\int\int\int_{V_{tot}} \frac{T_{ij}}{r}\mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_i}\int\int_S l_j\left(\rho v_i v_j + p_{ij}\right)\frac{\mathrm{d}S(\mathbf{y})}{r} - \int\int_S l_i \frac{\partial}{\partial t}(\rho v_i)\frac{\mathrm{d}S(\mathbf{y})}{r}. \tag{22}$$

For the solid immoveable surface, S, where the normal velocity of the fluid is zero, the above equation turns into the following, which constitutes Curle's fundamental result [2]:

$$4\pi c_0^2 \rho'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j}\int\int\int_{V_{tot}} \frac{T_{ij}}{r}\mathrm{d}\mathbf{y} - \frac{\partial}{\partial x_i}\int\int_S \frac{P_i}{r}\mathrm{d}S(\mathbf{y}), \tag{23}$$

where $P_i = -l_j p_{ij}$ is the ith component of the force per unit area exerted on the fluid by the rigid surface, S, and T_{ij} and P_i are taken at retarded times, $t - r/c_0$. According to Eq. (23), the sound radiated by a fluid flow in the presence of a rigid immoveable object, consists of two fields: (a) the field of the volume distribution of quadrupoles of strength, T_{ij}, per unit volume, and (b) the field of the surface distribution of dipoles of strength, P_i, per unit area.

Ffowcs Williams and Hawkings [3] extended Curle's theory to the case where the rigid object is moving. They derived an equation that takes the following form for a flow with small Mach numbers:

$$4\pi c_0^2 \rho'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j}\int\int\int_{V_{tot}} \frac{T_{ij}}{r}\mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_i}\int\int_S l_j p_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r} - \frac{\partial}{\partial t}\int\int_S \rho_0 U_n \frac{\mathrm{d}S(\mathbf{y})}{r}, \tag{24}$$

where U_n is the normal velocity of the rigid boundary.

3.5. Discontinuity of Lighthill's stress tensor on the rigid surface and its influence on the evaluation of Lighthill's integral

As shown in Section 3.3, the application of the divergence theorem to a vector field, $\mathbf{F}$, depends on the continuity of $\mathbf{F}$ on the surface of integration, which is the surface S in the case under consideration. It can be concluded that the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, determined by Eqs. (15) and (16), are discontinuous on S. On the one hand, on the exterior side of S the stress tensor T_{ij} is not zero; therefore in general, both vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are different from zero. On the other hand, on the interior side of S $T_{ij} = 0$ due to the rigidity of the object. Consequently, both vector fields, $\mathbf{F}_1$ and $\mathbf{F}_2$, are equal to zero. Thus, $\mathbf{F}_1$ and $\mathbf{F}_2$ have a break on the rigid surface, S, which is equal to the values of these vectors on the exterior side of S. The surface divergence of $\mathbf{F}_1$ and $\mathbf{F}_2$ on the surface, S, in accordance with Eq. (9), is determined as follows:

$$\nabla_S \cdot \mathbf{F}_1(\mathbf{y}) = -\mathbf{n} \cdot \mathbf{F}_1\left(\mathbf{y}^{(-)}\right), \tag{25}$$

$$\nabla_S \cdot \mathbf{F}_2(\mathbf{y}) = -\mathbf{n} \cdot \mathbf{F}_2\left(\mathbf{y}^{(-)}\right), \tag{26}$$

where the positive side of the surface S is the *internal* side since the normal vector, **n**, is directed outwards from the fluid. After the substitution of Eqs. (15) and (16), determining the vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, Eqs. (27) and (28) can be written as

$$\nabla_S \cdot \left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) = -l_i \frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}, \tag{27}$$

$$\nabla_S \cdot \left(T_{ij}\frac{1}{r}\right) = -l_j T_{ij}\frac{1}{r}, \tag{28}$$

As a result, the divergence theorem must be applied to the fields, $\mathbf{F}_1$ and $\mathbf{F}_2$, in its modified form (10), rather than in its traditional form (8), with the volume V_0 as the volume of integration. This leads to the following equations, which take place of Eqs. (11) and (12):

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int\int_{V_0} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} + \int\int_S \nabla_S \cdot \left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}S(\mathbf{y}), \tag{29}$$

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = \int\int\int_{V_0} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} + \int\int_S \nabla_S \cdot \left(T_{ij}\frac{1}{r}\right) \mathrm{d}S(\mathbf{y}). \tag{30}$$

If Eqs. (17), (18), (27) and (28) are substituted into the above equations, the following equations are obtained:

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = 0, \tag{31}$$

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right) \mathrm{d}\mathbf{y} = 0. \tag{32}$$

The substitution of the above equations to Eqs. (5) and (6) lead to the conclusion that Lighthill's transformation of the volume integral in the absence of solid boundaries remains valid in the flow where solid boundaries are present:

$$\int\int\int_{V_{tot}} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}\frac{\mathrm{d}\mathbf{y}}{r} = \frac{\partial^2}{\partial x_i \partial x_j}\int\int\int_{V_{tot}} \frac{T_{ij}}{r} \mathrm{d}\mathbf{y}. \tag{33}$$

Thus, the expression for the amplitude of density fluctuations, ρ', of the sound wave radiated by the fluid containing the rigid object, takes the following form:

$$\rho'(\mathbf{x}, t) = \frac{1}{4\pi c_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int\int\int_{V_{tot}} \frac{T_{ij}}{r} \mathrm{d}\mathbf{y} + \frac{1}{4\pi}\int\int_S \left(\frac{1}{r}\frac{\partial \rho'}{\partial n} + \frac{1}{r^2}\frac{\partial r}{\partial n}\rho' + \frac{1}{c_0 r}\frac{\partial r}{\partial n}\frac{\partial \rho'}{\partial t}\right) \mathrm{d}S(\mathbf{y}), \tag{34}$$

which represents the contribution of the current paper.

3.6. An alternative way of the evaluation of the volume integral

It may be noted that Eq. (33), leading to the solution, Eq. (34), can be obtained also without considering the layer of surface divergence. Indeed, since the surface S_1 encloses all regions where $T_{ij} \neq 0$, the volume integrals can be described by the integrals over S_1 *only*. In this case Eqs. (29) and (30) become

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_i}\left(\frac{\partial T_{ij}}{\partial y_j}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \int\int_{S_1} l_i \frac{\partial T_{ij}}{\partial y_j}\frac{\mathrm{d}S(\mathbf{y})}{r}, \tag{35}$$

$$\int\int\int_{V_{tot}} \frac{\partial}{\partial y_j}\left(T_{ij}\frac{1}{r}\right)\mathrm{d}\mathbf{y} = \int\int_{S_1} l_j T_{ij}\frac{\mathrm{d}S(\mathbf{y})}{r}. \tag{36}$$

As $T_{ij} = 0$ on S_1, the integrals in the right-hand of Eqs. (35) and (36) are zeros, which leads to Eqs. (31) and (32) and, in turn, to Eqs. (33) and (34). This procedure is considered in detail in Ref. [9].

4. Comparison of the obtained solution with Curle's equation

4.1. Comparison of the acoustic source terms

Evaluation of Eq. (34) and Curle's solution (22) shows that both solutions contain the volume integral over Lighthill's quadrupole sources. Both equations also contain surface integrals that describe the sound radiation produced by a layer of monopoles and a layer of dipoles on the surface, S. However, the strength of these layers is determined differently.

Let the comparison of the two solutions be made for the linear approximation, so that the terms of order, $v_i v_k$, can be neglected. In Curle's equation (22), the strength of the monopole layer is described by the *total* normal velocity on the surface, which is zero for a rigid immoveable object, while in Eq. (34) obtained here, the strength of the monopole layer is determined by the normal derivative of the fluid density in the acoustic wave, radiated or scattered by the rigid object. In other terms, the monopole term is determined by the normal component of the *acoustic scattered* velocity $\mathbf{v}_{sc}$.

This result leads to the significant conclusion that monopole acoustic sources may exist on the surface of an infinitely rigid immoveable object. For example, if there is a rotational (solenoidal) velocity field or an external source of acoustic waves, the boundary conditions on the surface of the object will require that the *total* normal velocity be equal to zero. At the same time, the scattered velocity, $\mathbf{v}_{sc}$, may be different from zero, because it must satisfy the boundary condition in combination with other velocity fields. These monopole sources can, apparently, produce sound of higher multipoles, including sound with dipole directivity patterns, if they are in different phases on opposite sides of the rigid object.

The strength of the dipole layer is also different in Curle's equation (22) and in Eq. (34) obtained here. According to Curle, the strength of the dipole layer is described by the *total* force acting upon the fluid (which includes, by means of the tensor p_{ij}, viscous forces), while in Eq. (34), the strength of the dipole layer is determined by the density fluctuations in the scattered wave on the surface. In other terms, the strength of the layer of dipoles on the surface is determined by the *scattered acoustic* component of the force, caused by density fluctuations in the scattered wave.

4.2. Applications of the obtained solution and Curle's equation to two well-known sound scattering and radiation problems

To demonstrate how the differences between Curle's equation (22) and the obtained Eq. (34) reveal in real-life applications, both equations are applied in this section to two simple acoustic problems, the solutions of which are well known. These problems have been chosen for the purpose of making obvious that Curle's equation gives predictions for the amplitude of radiated sound, which are different from known results, while predictions of the obtained equation coincide with results known from literature. Application of the obtained equation to more complex acoustic problems will be the subject of further publications.

4.2.1. Example 1. Sound scattering by a rigid sphere

Let Curle's solution (23) be applied to a sound scattering problem; for example, to the well-known case of plane wave scattering by a rigid immoveable sphere. This situation is described by Eq. (23) if the volume, V_0, where $T_{ij} \neq 0$, is small and its distance from the solid object is large in comparison with the acoustic wavelength. In these circumstances, the sound radiated by the quadrupole sources in V_0 can be considered as a plane wave near the solid object, and the problem under consideration reduces to a problem of sound scattering. To avoid unnecessary complexity, the rigid object is assumed to be a sphere of radius R_0.

For a single frequency acoustic wave, with temporal dependence $e^{-i\omega t}$, the expression for the density fluctuations in the scattered wave, determined by the second term in Curle's equation (23), can be written as follows:

$$4\pi c_0^2 \rho'_{Curle}(\mathbf{x}) = -\nabla \cdot \int\int_S \mathbf{P}(\mathbf{y}) \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \, dS(\mathbf{y}), \tag{37}$$

where $\mathbf{P}$ is the total force per unit area, $\mathbf{x} = (r_x, \theta_x, \varphi_x)$ is the radius vector of the observation point, $\mathbf{y} = (r_y, \theta_y, \varphi_y)$ is the radius vector of a source point on the surface, S, of the sphere, and (r, θ, φ) are the spherical co-ordinates with the origin in the centre of the sphere.

If the incident plane wave with velocity amplitude, U_0, approaches the solid sphere from the direction $\theta = \pi$, the pressure, P_{inc}, in the incident plane wave can be written as

$$P_{inc}(r_x, \theta_x) = \rho_0 c_0 U_0 e^{ikr_x \cos\theta_x}. \tag{38}$$

The exact solution for the pressure in the scattered wave, $P_{sc}(r_x, \theta_x)$, takes the form of a series of spherical Hankel functions of the first kind and Legendre polynomials [10,11]. In the case of an acoustically small sphere, for which $kR_0 \ll 1$, $P_{sc}(r_x, \theta_x)$ can be shown to take the following form on the surface of the sphere:

$$P_{sc}(R_0, \theta_x) = \tfrac{1}{2}\rho_0 c_0 U_0 k R_0 \, i \cos\theta_x + O\big((kR_0)^2\big). \tag{39}$$

Thus, the total acoustic pressure, $P_{tot} = P_{inc} + P_{sc}$, at $kR_0 \ll 1$ can be written as follows:

$$P_{tot}(R_0, \theta_x) = \rho_0 c_0 U_0 \big(1 + \tfrac{3}{2} ikR_0 \cos\theta_x\big). \tag{40}$$

Expanding the expression $\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$ into Taylor series and neglecting terms of order $(kR_0)^2$ and higher, it can be shown that Eq. (37) can be reduced to

$$\rho'_{Curle}(\mathbf{x}) = -\frac{\rho_0 U_0}{4\pi c_0}\frac{\partial}{\partial z_x}\left[\frac{\mathrm{e}^{\mathrm{i}kr_x}}{r_x}\int\int_S\left(\cos\theta_y + \frac{3}{2}\,\mathrm{i}kR_0\cos^2\theta_y\right)(1-\mathrm{i}kR_0\cos\alpha)\,\mathrm{d}S\right], \quad (41)$$

where $z_x = r_x \cos\theta_x$ and α is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$, determined by

$$\cos\alpha = \cos\theta_x\cos\theta_y + \sin\theta_x\sin\theta_y\cos\left(\varphi_x - \varphi_y\right). \quad (42)$$

Calculation of the surface integral in Eq. (41) leads to the following equation for Curle's prediction for the amplitude of the acoustic wave scattered by the sphere:

$$\rho'_{Curle}(\mathbf{x}) = -\frac{\rho_0 U_0 \omega^2 R_0^3}{c_0^3}\frac{\mathrm{e}^{\mathrm{i}kr_x}}{r_x}\left(-\frac{1}{2}\cos\theta_x + \frac{1}{3}\cos^2\theta_x\right). \quad (43)$$

Now Eq. (34), which was derived here, will be applied to the same problem of sound scattering by a small rigid sphere. The scattered acoustic wave, determined by Eq. (34), is the sum of a field, $\rho'_{mon}(\mathbf{x})$, radiated by a layer of monopoles, and a field, $\rho'_{dip}(\mathbf{x})$, radiated by a layer of dipoles on the surface of the sphere:

$$\rho'_{sc}(\mathbf{x}) = \rho'_{mon}(\mathbf{x}) + \rho'_{dip}(\mathbf{x}). \quad (44)$$

For a single frequency incident wave, the equations describing the two fields can be rewritten as follows:

$$\rho'_{mon}(\mathbf{x}) = \frac{1}{4\pi}\int\int_S \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\frac{\partial\rho'_{sc}}{\partial n}\,\mathrm{d}S(\mathbf{y}), \quad (45)$$

$$\rho'_{dip}(\mathbf{x}) = -\frac{1}{4\pi}\int\int_S \rho'_{sc}\left(\nabla_y\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\right)\mathbf{n}(\mathbf{y})\,\mathrm{d}S(\mathbf{y}). \quad (46)$$

For $kR_0 \ll 1$, the density fluctuations and their normal derivative on the surface S can be determined by the following equations:

$$\frac{\partial\rho'_{sc}}{\partial n} = \mathrm{i}\omega\,\frac{\rho_0}{c_0^2}\,U_0(1+\mathrm{i}kR_0\cos\theta)\cos\theta + O\big((kR_0)^2\big), \quad (47)$$

$$\rho'_{sc}(R_0) = \frac{1}{2}\frac{\rho_0}{c_0}\,U_0 kR_0\,\mathrm{i}\cos\theta + O\big((kR_0)^2\big). \quad (48)$$

Substitution of Eqs. (47) and (48) into Eqs. (45) and (46) for $kR_0 \ll 1$ gives:

$$\rho'_{mon}(\mathbf{x}) = \frac{\mathrm{i}\omega\rho_0 U_0}{4\pi c_0^2}\frac{\mathrm{e}^{\mathrm{i}kr_x}}{r_x}\int\int_S\big(1+\mathrm{i}kR_0\cos\theta_y\big)(1-\mathrm{i}kR_0\cos\alpha)\cos\theta_y\,\mathrm{d}S, \quad (49)$$

$$\rho'_{dip}(\mathbf{x}) = \frac{\mathrm{i}\omega\rho_0 U_0}{8\pi c_0^2}\frac{\mathrm{e}^{\mathrm{i}kr_x}}{r_x}\,kR_0\int\int_S(1-\mathrm{i}kR_0\cos\alpha)\cos\theta_y\cos\alpha\,\mathrm{d}S, \quad (50)$$

where the angle α is determined by Eq. (42). If only the most significant terms with respect to kR_0 and $1/r$ are retained in the integrals of Eqs. (49) and (50), the following expressions for $\rho'_{mon}(\mathbf{x})$, $\rho'_{dip}(\mathbf{x})$, and $\rho'_{sc}(\mathbf{x})$ can be obtained:

$$\rho'_{mon}(\mathbf{x}) = -\frac{\rho_0 U_0 \omega^2 R_0^3}{3c_0^3} \frac{e^{ikr_x}}{r_x} (1 - \cos\theta_x), \tag{51}$$

$$\rho'_{dip}(\mathbf{x}) = \frac{\rho_0 U_0 \omega^2 R_0^3}{6c_0^3} \frac{e^{ikr_x}}{r_x} \cos\theta_x, \tag{52}$$

$$\rho'_{sc}(\mathbf{x}) = \rho'_{mon}(\mathbf{x}) + \rho'_{dip}(\mathbf{x}) = -\frac{\rho_0 U_0 \omega^2 R_0^3}{3c_0^3} \frac{e^{ikr_x}}{r_x} \left(1 - \frac{3}{2}\cos\theta_x\right). \tag{53}$$

Comparison of Eq. (53) with Curle's prediction described by Eq. (43) shows that the dipole terms, proportional to $\cos\theta_x$ are identical in both equations. However, in Eq. (53), the dipole term is comprised of acoustic sources of two kinds: first, radiation generated by a surface distribution of density fluctuations, determined by $\rho'_{dip}(\mathbf{x})$, and second, the radiation generated by a surface distribution of normal velocity, determined by $\rho'_{mon}(\mathbf{x})$. The former sources are represented as a layer of elementary dipoles, or a double layer, while the latter sources are represented as a layer of elementary monopoles, or a single layer. The monopole sources generate the dipole sound due to different phases on the opposite hemispheres.

The main difference between Eq. (53) derived here and Curle's prediction expressed in Eq. (43) is the presence of the *monopole* term in the scattered field. Indeed, in Eq. (53) there is a term independent of the angle θ_x, while, according to Curle's theory, the lowest multipole in the sound generated by a rigid immoveable object is a *dipole*, which depends on the observation direction as $\cos\theta_x$. Comparison of the two equations with well-known results from textbooks; for example, from Refs. [12,13], leads to the conclusion that Curle's claim, that the monopole component is absent in the acoustic wave radiated by an absolutely rigid immoveable object, is incorrect. Eq. (53) coincides with equations for the amplitude of the scattered sound wave published in Refs. [12,13], while Curle's prediction (43) differs from these equations by the *absence of the monopole term*.

The second difference between Curle's equation (43) and the obtained Eq. (53) is the presence of a term proportional to $\cos^2\theta_x$. This term describes *quadrupole* sound and should not appear in this analysis, which is restricted to terms of order no higher than kR_0 [10].

4.2.2. Example 2. Sound generation by a rigid sphere in a variable velocity field

This section deals with a situation that is similar to sound scattering by a small rigid sphere considered in the previous example. The difference between the two cases is that in this situation the incident field is only a *velocity field*, so there are no pressure fluctuations in the incident field. This situation may occur, for example, when a rigid sphere is submerged in a flow of low Mach number and with a typical vortex size much larger than the diameter of the sphere. It is also important to note that the sound radiation by a sphere in a variable, spatially uniform, velocity field is equivalent to sound radiation by a sphere vibrating in the fluid which is at rest [13].

Both the obtained solution of Eqs. (44)–(46) and Curle's result of Eq. (37) can be applied to the situation under consideration in the same way as in the first example. The pressure fluctuations on the surface of the sphere are determined by the following equation [13]:

$$P_{tot}(R_0, \theta_x) = \tfrac{1}{2}\rho_0 c_0 U_0 k R_0 \text{ i} \cos\theta_x + O\big((kR_0)^2\big). \tag{54}$$

Substitution of Eq. (54) into Eq. (37) gives the following expression for the density fluctuations in the radiated sound wave on the basis of Curle's solution:

$$\rho'_{Curle}(\mathbf{x}) = -\frac{\rho_0 U_0}{4\pi c_0}\frac{\partial}{\partial z_x}\left[\frac{e^{ikr_x}}{r_x}\int\int_S\left(\frac{1}{2}\,\mathrm{i}kR_0\cos^2\theta_y\right)(1 - \mathrm{i}kR_0\cos\alpha)\,\mathrm{d}S\right], \tag{55}$$

where $z_x = r_x \cos\theta_x$ and $\cos\alpha$ is determined by Eq. (42). Further calculations lead to the following equation for Curle's prediction of the amplitude of density fluctuations in the sound wave radiated by a small solid sphere in a variable velocity field:

$$\rho'_{Curle}(\mathbf{x}) = -\frac{\rho_0 U_0 \omega^2 R_0^3}{c_0^3}\frac{e^{ikr_x}}{r_x}\left(-\frac{1}{6}\cos\theta_x + \frac{1}{3}\cos^2\theta_x\right). \tag{56}$$

The method of application of the obtained solutions (44)–(46) to the case under consideration is analogous to the method used in the first example. Due to the spatial homogeneity of the incident velocity field, the normal component of the fluid velocity of the radiated wave is determined by the following equation,

$$\frac{\partial\rho'_{sc}}{\partial n} = \mathrm{i}\omega\,\frac{\rho_0}{c_0^2}\,U_0\cos\theta_x, \tag{57}$$

which, as opposed to Eq. (47), has only a zero order term with respect to kR_0. The substitution of Eq. (57) into Eq. (45) gives

$$\rho'_{mon}(\mathbf{x}) = \frac{\mathrm{i}\omega\rho_0 U_0}{4\pi c_0^2}\frac{e^{ikr}}{r}\int\int_S(1 - \mathrm{i}kR_0\cos\alpha)\cos\theta_y\,\mathrm{d}S, \tag{58}$$

and, after simplification:

$$\rho'_{mon}(\mathbf{x}) = \frac{\rho_0 U_0\omega^2 R_0^3}{3c_0^3}\frac{e^{ikr_x}}{r_x}\cos\theta_x. \tag{59}$$

The contribution of the layer of dipoles, ρ'_{dip}, on the surface S is determined by the same equation (52) as in the first example.

The substitution of Eqs. (59) and (52) into Eq. (44) gives the following prediction for the amplitude of density fluctuations in the far field for an acoustic wave generated by a rigid sphere in a variable velocity field:

$$\rho'_{sc}(\mathbf{x}) = \frac{\rho_0 U_0\omega^2 R_0^3}{2c_0^3}\frac{e^{ikr_x}}{r_x}\cos\theta_x. \tag{60}$$

Ref. [13] states that the problem of sound radiation by a rigid sphere in a variable velocity field is equivalent to the problem of sound radiation by the sphere vibrating with equal amplitude, and the solution of the latter problem is shown in many textbooks, including Refs. [11,13]. The comparison of Eqs. (56) and (60) with the known solution demonstrates that the correct prediction for the amplitude of density fluctuations is given by Eq. (60), which has been derived from Eq. (34), obtained in this article, while Curle's prediction of Eq. (56) differs from the correct result by a factor of 3.

5. Conclusions

The derivation of Curle's equation for the amplitude of aerodynamic sound radiated by a rigid immovable object in a fluid flow [2] has been examined in detail. The analysis showed that in Curle's calculations, the influence of the discontinuity of the Lighthill's stress tensor, T_{ij}, on the rigid surface has been erroneously omitted.

Taking account of the surface discontinuity of Lighthill's stress tensor leads to an equation for the amplitude of the radiated sound wave, which differs from Curle's solution in two ways. First, the obtained equation determines that the radiated sound wave consists of the fields radiated by both *dipole and monopole* source distributions on the rigid surface, while, according to Curle, there is only a *dipole* source distribution on the surface of a rigid immoveable object. Second, the strength of the dipole sources in the obtained equation is determined only by the *scattered* component of pressure, which is proportional to the amplitude of density fluctuations in the scattered wave, while in Curle's solution the strength of the dipole sources is determined by the *total* pressure on the surface.

The current analysis shows that Curle's equation for the amplitude of a sound wave radiated by a rigid object in a fluid flow has been derived with significant mathematical inaccuracy. The equation is shown to give incorrect predictions for the well-known problems of sound scattering and radiation by a rigid sphere. In some situations, Curle's equation may give results that coincide with the correct ones; however, such coincidence is fortuitous. These conclusions apply also to the Ffowcs Williams and Hawkings equation, which for an immoveable rigid object reduces to the equation derived by Curle.

Acknowledgements

The present work has been supported by a Large Grant from the Australian Research Council.

References

[1] M.J. Lighthill, On sound generated aerodynamically, Proceedings of the Royal Society A 211 (1952) 564–586.
[2] N. Curle, The influence of solid boundaries upon aerodynamic sound, Proceedings of the Royal Society A 231 (1955) 505–514.
[3] J.E. Ffowcs Williams, D.L. Hawkings, Sound generation by turbulence and surfaces in arbitrary motion, Philosophical Transactions of the Royal Society A 264 (1969) 321–342.
[4] D.A. Bies, J.M. Pickles, D.J.J. Leclercq, Aerodynamic noise generation by a stationary body in a turbulent air stream, Journal of Sound and Vibration 204 (1997) 631–643.
[5] A.T. Fedorchenko, On some fundamental flaws in present aeroacoustic theory, Journal of Sound and Vibration 232 (2000) 719–782.
[6] O.D. Kellogg, Foundations of the Potential Theory, Verlag von Julius Springer, Berlin, 1929.

[7] N. Kemmer, Vector Analysis. A Physicist's Guide to the Mathematics of Fields in Three Dimensions, Cambridge University Press, London, 1977.
[8] G.A. Korn, T.M. Korn, Mathematical Handbook for Scientists and Engineers, 2nd Edition, McGraw Hill Book Company, New York, 1971.
[9] A. Zinoviev, On the use of the divergence theorem in the derivation of Curle's formula for the amplitude of aerodynamic sound, Proceedings of the 17th International Congress on Acoustics, Rome, Italy, September 2–7, 2001.
[10] J.J. Bowman, T.B.A. Senior, P.L.E. Uslenghi, Electromagnetic and Acoustic Scattering by Simple Shapes, Hemisphere Publishing Corporation, New York, 1990.
[11] P.M. Morse, Vibration and Sound, 2nd Edition, McGraw Hill Book Company, New York, 1948.
[12] A.D. Pierce, Acoustics: An Introduction to Its Physical Principles and Applications, Acoustical Society of America, New York, 1989.
[13] L.D. Landau, E.M. Lifshitz, Fluid Mechanics, Course of Theoretical Physics, Vol. 6, Pergamon Press, Oxford, 1959.